A SHORT PROOF OF MONOTONICITY OF A FUNCTION INVOLVING THE PSI AND EXPONENTIAL FUNCTIONS

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Abstract. In the short note, a simple proof is provided for the increasing monotonicity of the function $\psi(x)+\ln\bigl(e^{1/x}-1\bigr)$ on $(0,\infty)$, where $\psi(x)$ is the well-known psi function.

It is well-known that the classical gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \tag{1}$$

for $x>0$, the psi function $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, and the polygamma functions $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ play central roles in the theory of special functions and have much extensive applications in many branches.

In [4, Theorem 2], it was discovered that if $a\le-\ln2$ and $b\ge0$, then

$$a-\ln\bigl(e^{1/x}-1\bigr)<\psi(x)<b-\ln\bigl(e^{1/x}-1\bigr) \tag{2}$$

holds for $x>0$.

In [3, Theorem 2.8], the inequality (2) was sharpened as follows: If $a\le-\gamma$ and $b\ge0$, then the inequality (2) is valid for $x>0$, where the constants $-\gamma=-0.577\dots$ (the negative of Euler-Mascheroni's constant) and 0 are the best possible.

In [2], the function

$$\phi(x)=\psi(x)+\ln\bigl(e^{1/x}-1\bigr) \tag{3}$$

was proved to be strictly increasing on $(0,\infty)$ and

$$\lim_{x\to\infty}\phi(x)=0. \tag{4}$$

In [9], among other things, the function $\phi(x)$ was proved to be not only strictly increasing but also strictly concave on $(0,\infty)$, with the limits $\lim_{x\to0^+}\phi(x)=-\gamma$ and (4).

In [2, 9], the proofs of the increasing monotonicity of $\phi(x)$ spent respectively almost two printed pages.

The aim of this short note is to provide a simple proof for the increasing monotonicity of the function $\phi(x)$ as follows.

Theorem 1. *The function $\phi(x)$ is strictly increasing on $(0,\infty)$.*

Proof. It is well-known that

$$\Gamma(x+1)=x\Gamma(x) \tag{5}$$

for $x>0$. Taking the logarithm on both sides of the above equation and differentiating yields

$$\psi(x+1)=\psi(x)+\frac{1}{x}. \tag{6}$$

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Therefore, the exponential function of $\phi(x)$ satisfies

$$e^{\phi(x)} = e^{\psi(x)}\bigl(e^{1/x} - 1\bigr) = e^{\psi(x)+1/x} - e^{\psi(x)} = e^{\psi(x+1)} - e^{\psi(x)} \triangleq f(x),$$
$$f'(x) = e^{\psi(x+1)}\psi'(x+1) - e^{\psi(x)}\psi'(x) \triangleq h(x+1) - h(x),$$
$$h'(x) = [e^{\psi(x)}\psi'(x)]' = e^{\psi(x)}\bigl\{\psi''(x) + [\psi'(x)]^2\bigr\}.$$

In [1, p. 208] and [4, Lemma 1.1], the inequality

$$[\psi'(x)]^2 + \psi''(x) > 0 \tag{7}$$

for $x \in (0,\infty)$ was verified. Hence, the function $h(x)$ is strictly increasing, and so $f'(x) > 0$ on $(0,\infty)$. As a result, the function $f(x)$, and then $\phi(x)$, is strictly increasing on $(0,\infty)$. □

Remark 1. It is well-known that the n-th harmonic numbers are defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{8}$$

for $n \in \mathbb{N}$ and that H_n can be expressed in terms of the psi function $\psi(x)$ by

$$H_n = \psi(n+1) + \gamma. \tag{9}$$

Consequently, the increasing monotonicity of $\phi(x)$ implies the sharp double inequality in [3, Theorem 2.8] and the sharp inequalities for harmonic numbers H_n in [2, pp. 386–387]: For $n \in \mathbb{N}$,

$$1 + \ln\bigl(\sqrt{e} - 1\bigr) - \ln\bigl(e^{1/(n+1)} - 1\bigr) \le H_n < \gamma - \ln\bigl(e^{1/(n+1)} - 1\bigr). \tag{10}$$

The constants $1 + \ln\bigl(\sqrt{e} - 1\bigr)$ and γ in (10) are the best possible.

Some sharp inequalities for harmonic numbers were also established in [5, 6, 7, 8] and related references therein.

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